Repeated Dimensions of Semisimple Lie Algebra Representations

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Abstract

We will explore sequences of repeated dimensions of finite dimensional representations of non-isomorphic non-dual semisimple Lie algebras over the complex numbers. We give an explicit construction showing that every semisimple Lie algebra of rank 2 or higher admits such a sequence with infinitely many elements. By constructing an elliptic curve birational with the Weyl dimension formula for representations of the Lie algebra of type A_2, we show that A_2 admits an infinite sequence of dimensions shared by m non-isomorphic non-dual irreducible representations, for any $m > 0$. Then we will show that A_2 is the only rank 2 simple Lie algebra such that this property for arbitrary m. We will conclude by showing that the existence of such a sequence for arbitrary m of higher rank simple Lie algebras is contingent on the Bombieri-Lang conjecture.

1 Introduction

1.1 Overview

This paper addresses the question of which semisimple Lie groups have non-dual non-isomorphic irreducible representations of the same dimension. The only discussion about this which the author is aware of is a special case of this question on mathoverflow.net [1], although no answer was presented there. In this paper we call two representations dual if they are equivalent via a symmetry of the Dynkin diagram. For example, for type A_n, $n \geq 2$, any representation will have the same dimension if we perform the involution of swapping simple roots $\alpha_i \leftrightarrow \alpha_{n-i}$. This trivializes the problem for any representation when the Dynkin diagram contains a symmetry such as the above, so we will treat dual representations as the same. In light of this, we can prove that there exist infinitely many pairs of non-isomorphic irreducible representations of equal dimension for any rank ≥ 2 semisimple Lie algebra without relying on duality arguements.

1.2 Definitions and Notation

Let $\mathfrak{g}$ be a semisimple Lie algebra over the complex numbers, and let $\mathfrak{g}_1$, $\mathfrak{g}_2$, $\ldots \mathfrak{g}_m$ be the simple direct factors of $\mathfrak{g}$. Let V be an irreducible representation of $\mathfrak{g}$ of highest weight λ. Then we have a tensor decomposition

$$V_\lambda = V_{\lambda_1} \otimes V_{\lambda_2} \otimes \ldots \otimes V_{\lambda_m}$$

Let Φ denote the root system of $\mathfrak{g}$ with positive roots Φ^+, and Φ_i denote the root system of $\mathfrak{g}_i$ with positive roots Φ_i^+. A corollary to the Weyl character formula, known as the Weyl dimension formula, states that the dimensions of V_λ is given by

$$\text{dim}V_\lambda = \frac{\prod_{\alpha\in\Phi^+}(\lambda+\rho,\alpha)}{\prod_{\alpha\in\Phi^+}(\rho,\alpha)}$$

for ρ half the sum of positive roots of Φ and roots α. A full derivation can be found here [2]. We can decompose this into the product of dimensions of the V_i as follows (with ρ_i half the sum of positive roots of Φ_i):

$$\text{dim}V_\lambda = \frac{\prod_{\alpha\in\Phi^+}(\lambda+\rho,\alpha)}{\prod_{\alpha\in\Phi^+}(\rho,\alpha)} = \prod_{i=1}^{m} \frac{\prod_{\alpha\in\Phi_i^+}(\lambda_i+\rho_i,\alpha)}{\prod_{\alpha\in\Phi_i^+}(\rho_i,\alpha)} = \prod_{i}^{m} \text{dim}V_{\lambda_i}$$

where each V_i has highest weight λ_i. Thus, the dimension of a representation of a semisimple Lie algebra is entirely determined by the dimensions of its substituent simple Lie algebra representations. Because of this, for the rest of the paper $\mathfrak{g}$ will be understood to denote a simple Lie algebra unless otherwise stated. Dropping the subscripts of the λ_i's, we turn our attention to some comments on notation.

For a simple Lie algebra $\mathfrak{g}$ of rank n an irreducible representation V may be uniquely determined by its highest weight λ by the theorem of the highest weight[3]. (This justifies our denotation of irreducible representations by their highest weights as V_λ.) The weights themselves may be written uniquely as an integral combination of fundamental weights $\omega_1, \ldots, \omega_n$, so we will write

$$\lambda = \sum_{i=1}^{n} a_i\omega_i$$

for some fundamental weight labels $a_i \in \mathbb{Z}$. As highest weights lie in the fundamental Weyl chamber, weight labels are taken to be non-negative when describing a highest weight, but may be negative for other weights. For convenience, λ may be denoted as a vector:

$$\lambda = (a_1, \ldots, a_n)$$

where the a_i are the same fundamental weight labels as above.

1.3 Some Useful Theorems

For a simple Lie algebra $\mathfrak{g}$ we define a function $f_{\mathfrak{g}}$ which contains all of the information of $\dim V_\lambda$ but has the added property that it is homogeneous and is defined on all of $\mathbb{R}$:

$$f_{\mathfrak{g}} : \mathbb{R}^n \mapsto \mathbb{R}$$

$$f_{\mathfrak{g}}(a_1, a_2, \ldots, a_n) = \frac{\prod_{\alpha\in\Phi^+} \sum_{i=1}^n (a_i\omega_i, \alpha)}{\prod_{\alpha\in\Phi^+} (\rho, \alpha)}$$

Observe that when $\{a_1, \ldots, a_n\} \in \mathbb{Z}^n$ this equals $\dim V_{\lambda-1}$ where $\lambda = \sum_{i=1}^n a_i\omega_i$:

$$\begin{aligned} f(a_1, \ldots, a_n) &= \frac{\prod_{\alpha\in\Phi^+} \sum_{i=1}^n (a_i\omega_i, \alpha)}{\prod_{\alpha\in\Phi^+} (\rho, \alpha)} && (1)\\ &= \frac{\prod_{\alpha\in\Phi^+} (\sum_{i=1}^n (a_i - 1)\omega_i + \rho, \alpha)}{\prod_{\alpha\in\Phi^+} (\rho, \alpha)} && (2)\\ &= \frac{\prod_{\alpha\in\Phi^+} (\lambda - 1 + \rho, \alpha)}{\prod_{\alpha\in\Phi^+} (\rho, \alpha)} && (3)\\ &= \dim V_{\lambda-1} && (4) \end{aligned}$$

where going between steps 1 and 2 we used the identity

$$\rho = \sum_{i=1}^n \omega_i$$

The subscript of f may be omitted when $\mathfrak{g}$ is unambiguous. In a minor abuse of notation when λ is unambiguous we may write $f(a_1, \ldots, a_n)$ as simply $f(\lambda)$.

To prove that f is homogeneous, we let $c \in \mathbb{N}$ and compute $f(ca_1, \ldots, ca_n)$:

$$\begin{aligned} f(ca_1, \ldots, ca_n) &= \frac{\prod_{\alpha\in\Phi^+} \sum_{i=1}^n (ca_i\omega_i, \alpha)}{\prod_{\alpha\in\Phi^+} (\rho, \alpha)}\\ &= c^n \frac{\prod_{\alpha\in\Phi^+} \sum_{i=1}^n (a_i\omega_i, \alpha)}{\prod_{\alpha\in\Phi^+} (\rho, \alpha)}\\ &= c^n f(a_1, \ldots, a_n) \end{aligned}$$

We are now ready to proceed with our first theorem.

Theorem 1. *Let $\mathfrak{g}$ be a simple Lie algebra. Suppose for some $m \in \mathbb{N}$ there exist a set of m irreducible representations of $\mathfrak{g}$ with distinct highest weights $\{\lambda_1, \ldots, \lambda_m\}$ and such that*

$$f(\lambda_1) = f(\lambda_2) = \ldots = f(\lambda_m)$$

Then there exist infinitely many such sets of m such elements.

Proof. Due to the homogeneity of f, for any $c \in \mathbb{N}$ we have

$$f(\lambda_i) = f(\lambda_j) \implies f(c\lambda_i) = f(c\lambda_j)$$

for $i, j \in \{1, 2, \ldots, m\}$. Thus, we obtain infinitely many lists (one for each $c \in \mathbb{N}$) $\{c\lambda_1, \ldots, c\lambda_m\}$ such that

$$f(c\lambda_1) = f(c\lambda_2) = \ldots = f(c\lambda_m)$$

□

Our next theorem concerns the importance of f evaluated at points of $\mathbb{Q}^n$:

Theorem 2. *Let $\lambda_1, \lambda_2 \in \mathbb{Q}^n$ with $\lambda_1 \neq \lambda_2$, and suppose*

$$f(\lambda_1) = f(\lambda_2)$$

Then there exist some ${\lambda'}_1, {\lambda'}_2 \in \mathbb{Z}^n, {\lambda'}_1 \neq {\lambda'}_2$ such that

$$f({\lambda'}_1) = f({\lambda'}_2)$$

Moreover, for some $m \in \mathbb{Z}$,

$${\lambda'}_i = m\lambda_i$$

for $i = 1, 2$.

Proof. For $i = 1, 2$ we know that $\lambda_i \in \mathbb{Q}$, thus $\lambda_i = (a_{i_1}, \ldots, a_{i_n})$ where each $a_{i_j} = p_{i_j}/q_{i_j}$ for unique $p_{i_j}, q_{i_j} \in \mathbb{Z}$, $q_{i_j} > 0$ such that $\gcd(p_{i_j}, q_{i_j}) = 1$. Define

$$m = \prod_{i=1}^{2} \prod_{j=1}^{n} q_{i_j}$$

Then $a_{i_j} m \in \mathbb{Z}$ (since $q_{i_j} | m_i$), thus $m\lambda_i \in \mathbb{Z}^n$. Then by the homogeneity of f we must have $f(m\lambda_1) = f(m\lambda_2)$, and we are done.

□

2 General Result

We state a result for simple Lie algebras below. Later in this section we will state a result for semisimple Lie algebras.

Theorem 3. *Every compact simple Lie algebra of rank 2 or higher admits infinitely many pairs of non-dual non-isomorphic irreducible representations of equal dimension.*

We prove this on a case by case basis.

2.1 Infinite Families

2.1.1 $A_n, n \geq 2$

The dimension formula for A_1 is $\dim V_\lambda = \lambda + 1$, hence each irreducible representation of A_1 has a unique dimension. Thus, it is only possible to have repeated dimensions in A_n for $n \geq 2$.

We will use [3] for many of the following definitions. The standard basis for simple roots of A_n is $\alpha_i = e_i - e_{i+1}, 1 \leq i \leq n$ (where e_i is the usual basis vector of all zeroes except a 1 in the ith entry). Observe that the simple roots of A_n lie in $\mathbb{R}^{n+1}$, not $\mathbb{R}^n$. Thus, the roots are all vectors in $\mathbb{Z}^n$ with a single 1 and a single -1 entry, the rest zeroes. For our positive roots let us choose roots $\alpha \in \Phi$ such that the first nonzero entry of α is positive. Then each positive root $\alpha \in \Phi^+$ is of the form $\alpha = e_i - e_j$ for some $1 \leq i \leq j \leq n+1$.

Then we have

$$f_{A_n}(\lambda) = \frac{\prod_{\alpha\in\Phi^+} \sum_{i=1}^{n} (a_i\omega_i, \alpha)}{\prod_{\alpha\in\Phi^+}(\rho, \alpha)}$$

To simplify, we note the fundamental weights ω_i are

$$\begin{aligned}\omega_i =& \frac{1}{n+1}[(n-i+1)\alpha_1 + 2(n-i+1)\alpha_2 + \ldots \\ &+ (i-1)(n-i+1)\alpha_{i-1} + i(n-i+1)\alpha_i + i(n-1)\alpha_{i+1} + \ldots + i\alpha_n]\end{aligned}$$

Recall that the fundamental weights are defined as the vectors satisfying

$$2\frac{(\omega_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}$$

for simple roots α_j. Then because $(\alpha_j, \alpha_j) = 2$, we obtain $(\omega_i, \alpha_j) = \delta_{ij}$, the Kronecker delta.

Next we compute the inner product of these positive roots with the fundamental weights.

First note that

$$\begin{aligned}\alpha &= e_i - e_j \\ &= e_i - e_{i+1} + e_{i+1} - e_{i+2} + \ldots + e_{j-2} - e_{j-1} + e_{j-1} - e_j \\ &= (e_i - e_{i+1}) + (e_{i+1} - e_{i+2}) + \ldots + (e_{j-2} - e_{j-1}) + (e_{j-1} - e_j) \\ &= \alpha_i + \alpha_{i+1} + \ldots + \alpha_{j-2} + \alpha_{j-1} \\ &= \sum_{k=i}^{j-1} \alpha_k\end{aligned}$$

So we can decompose the inner product (α, ω_l) as

$$\begin{aligned}(\alpha, \omega_l) &= (e_i - e_j, \omega_l) \\ &= (\sum_{k=i}^{j-1} \alpha_k, \omega_l) \\ &= \sum_{k=i}^{j-1} \delta_{kl}\end{aligned}$$

Thus for $\alpha = e_i - e_j$ we get that $(\alpha, \omega_l) = \mathbb{1}[i \leq l < j]$ where $\mathbb{1}[\ldots]$ denotes the indicator function (if its argument is true then it outputs 1, else 0). Now we are ready to compute $f_{A_n}(\lambda)$. First, addressing the numerator of f we have

$$\begin{aligned}\prod_{\alpha \in \Phi^+} \sum_{l=1}^{n} (a_l \omega_l, \alpha) &= \prod_{i=1}^{n} \prod_{j=i+1}^{n+1} \sum_{l=1}^{n} (a_l \omega_l, e_i - e_j) \\ &= \prod_{i=1}^{n} \prod_{j=i+1}^{n+1} \sum_{l=1}^{n} a_l \mathbb{1}[i \leq l < j]\end{aligned}$$

We can understand what this expression means by computing it in reverse order, i.e. starting with $i = n$ and working down to $i = 1$.

The term involving $i = n, j = n + 1$ can be simply read off:

$$\sum_{l=1}^{n} a_l \mathbb{1}[n \leq l < n + 1] = a_n$$

Moreover, any terms of the form $j = i + 1$ is simply a_i. When $j = i + 2$ we get $a_i + a_{i+1}$, and so on. In the parlance of computer science we might call this the product of all (nonempty) substrings of $(a_1, a_2, \ldots, a_n)$, meaning it's the product of all combinations of the form $(a_i + a_{i+1} + \ldots + a_{j-1} + a_j)$. Thus we can rewrite the numerator as

$$\prod_{\alpha \in \Phi^+} \sum_{l=1}^{n} (a_l \omega_l, \alpha) = \prod_{1 \leq i \leq j \leq n} \sum_{l=i}^{j} a_l$$

Note that the denominator of f is the numerator evaluated at $\rho = \sum_{l=1}^{n} \omega_l = (1, 1, \ldots, 1)$. This is a consequence of the facts that for a given n the denominator is constant, and that the dimension of the trivial representation must always be 1. Evaluating this we find that the denominator must be equal to

$$\prod_{1 \leq i \leq j \leq n} \sum_{l=i}^{j} 1 = \prod_{i=1}^{n} i!$$

Thus the denominator is the superfactorial of n. Putting this together we have

$$f_{A_n}(\lambda) = \frac{\prod_{1 \leq i \leq j \leq n} \sum_{l=i}^{j} a_l}{\prod_{i=1}^{n} i!} \tag{5}$$

For the reader's convenience the first few terms of f_{A_n} are provided:

$$f_{A_1}(\lambda) = a_1$$

$$f_{A_2}(\lambda) = \frac{1}{2}(a_1 + a_2)a_1a_2$$

$$f_{A_3}(\lambda) = \frac{1}{12}(a_1 + a_2 + a_3)(a_1 + a_2)(a_2 + a_3)a_1a_2a_3$$

$$f_{A_4}(\lambda) = \frac{1}{288}(a_1+a_2+a_3+a_4)(a_1+a_2+a_3)(a_2+a_3+a_4)(a_1+a_2)(a_2+a_3)(a_3+a_4)a_1a_2a_3a_4$$

We will begin by considering λs which are a linear combination of at most two fundamental weights; this greatly simplifies calculations by giving us a nice closed form for $f_{A_n}(\lambda)$.

Suppose that for $i = 1, 2, \ldots, n-3$ we take a_i to be equal to one. To avoid constantly writing out a_{n-2} and a_{n-1} we will simply define a, b such that $a = a_{n-2}$ and $b = a_{n-1}$. Then $\lambda = (1, 1, \ldots, 1, a, b)$. Then by equation 5 we have

$$f_{A_n}(\lambda) = \frac{\prod_{1 \le i \le j \le n} \sum_{l=i}^{j} a_l}{\prod_{i=1}^{n} i!}$$

We will prove by induction that the following holds (for λ of the form above) for $n \ge 2$:

$$f(\lambda) = \frac{b(n+a+b-2)!(n+a-2)!}{n!(n-1)!(a-1)!(a+b-1)!} \tag{6}$$

Proof. Base case ($n = 2$):

$$f_{A_2}(\lambda) = \frac{\prod_{1 \le i \le j \le 2} \sum_{l=i}^{j} a_l}{\prod_{i=1}^{n} i!}$$

$$= \frac{(\sum_{l=1}^{1} a_l)(\sum_{l=2}^{2} a_l)(\sum_{l=1}^{2} a_l)}{2!}$$

$$= \frac{ab(a+b)}{2}$$

and

$$\frac{b(n+a+b-2)!(n+a-2)!}{n!(n-1)!(a-1)!(a+b-1)!}$$

$$= \frac{b(a+b)!(a)!}{2!(a-1)!(a+b-1)!}$$

$$= \frac{ab(a+b)}{2}$$

Thus the two expressions are equal for $n = 2$.

For the induction step, assume that for some $n \in \mathbb{N}$ equation 6 holds for all A_k with $k \leq n$, and we will prove that it holds for $k = n + 1$ as well.

First, we reindex the numerator product so that it more clearly resembles the form of A_n (that is, in the second step we now treat λ to look like ($a_0 = 1, a_1 = 1, \ldots, a_{n-1} = a, a_n = b$)):

$$\begin{aligned}
\frac{\prod_{1\leq i\leq j\leq n+1}\sum_{l=i}^{j} a_l}{\prod_{i=1}^{n+1} i!} &= \frac{\prod_{0\leq i\leq j\leq n}\sum_{l=i}^{j} a_l}{\prod_{i=1}^{n+1} i!} \\
&= \left(\prod_{i=0\leq j\leq n}\sum_{l=i}^{j} a_l\right)\left(\frac{(\prod_{1\leq i\leq j\leq n}\sum_{l=i}^{j} a_l)}{\prod_{i=1}^{n+1} i!}\right) \\
&= \left(\prod_{i=0\leq j\leq n}\sum_{l=i}^{j} a_l\right)\frac{f_{A_n}(\lambda)}{(n+1)!}
\end{aligned}$$

But we can break up the right hand side even more, using the fact that $a_l = 1$ for $l < n - 1$:

$$\begin{aligned}
\prod_{i=0\leq j\leq n}\sum_{l=i}^{j} a_l &= \left(\prod_{i=0\leq j\leq n-2}\sum_{l=i}^{j} a_l\right)\left(\sum_{l=i}^{n-1} a_l\right)\left(\sum_{l=i}^{n} a_l\right) \\
&= \left(\prod_{0\leq j\leq n-2}\sum_{l=0}^{j} 1\right)\left(a + \sum_{l=0}^{n-2} 1\right)\left(a + b + \sum_{l=0}^{n-2} 1\right) \\
&= \left(\prod_{0\leq j\leq n-2} j + 1\right)(a + n - 1)(a + b + n - 1) \\
&= (n-1)!(a + n - 1)(a + b + n - 1)
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\prod_{1\leq i\leq j\leq n+1}\sum_{l=i}^{j} a_l}{\prod_{i=1}^{n+1} i!} &= \frac{(n-1)!(a+n-1)(a+b+n-1)f_{A_n}(\lambda)}{(n+1)!} \\
&= \frac{(a+n-1)(a+b+n-1)}{n(n+1)} f_{A_n}(\lambda) \\
&= \frac{(a+n-1)(a+b+n-1)}{n(n+1)} \frac{b(n+a+b-2)!(n+a-2)!}{n!(n-1)!(a-1)!(a+b-1)!} \\
&= \frac{b(n+a+b-1)!(n+a-1)!}{(n+1)!n!(a-1)!(a+b-1)!}
\end{aligned}$$

which is what we wanted to prove, thus the induction step is complete. Having proven the base case and the induction hypothesis, we conclude that equation 6 is correct. □

Now we consider $\lambda_1 = (1,1,\ldots,1,1,b+1)$ and $\lambda_2 = (1,1,\ldots,1,a+1,b-a+1)$. Then

$$f(\lambda_1) = \frac{(b+n)!}{b!n!}$$
$$f(\lambda_2) = \frac{(b-a+1)(n+a-1)!(b+n)!}{a!(b+1)!(n-1)!n!}$$

Suppose we wanted to find a, b such that $f(\lambda_1) = f(\lambda_2)$. This would mean

$$\frac{(b+n)!}{b!n!} = \frac{(b-a+1)(n+a-1)!(b+n)!}{a!(b+1)!(n-1)!n!} \tag{7}$$

so

$$\frac{(b+1)!}{b!(b-a+1)} = \frac{(n+a-1)!}{a!(n-1)!}$$

hence

$$\frac{b+1}{b-a+1} = \binom{n+a-1}{a}$$

thus

$$1 + \frac{a}{b+1-a} = \binom{n+a-1}{a}$$

where $\binom{a}{b}$ denotes the binomial coefficient equal to $\dfrac{a!}{b!(a-b)!}$. Rearranging this, we find that equation 7 is satisfied when

$$b = \frac{a}{\binom{n+a-1}{a} - 1} + a - 1 = \frac{a(a!)(n-1)!}{(n+a-1)! - a!(n-1)!} + a - 1$$

for any $a \in \mathbb{N}$.

Noting that b is now a rational number and not necessarily an integer, applying the techniques of theorem 2 we can construct integral λs with equal dimension, given any natural number a:

$$\lambda_1 + 1 = (q, q, \ldots, q, q, p+q)$$

$$\lambda_2 + 1 = (q, q, \ldots, q, (a+1)q, p-(a-1)q)$$

where

$$p = a! + \frac{a(a+n-1)!}{(n-1)!} - \frac{(a+n-1)!}{(n-1)!} \tag{8}$$
$$q = \frac{(a+n-1)!}{(n-1)!} - a!$$

The following examples were computed in Sage. For A_{15}, for $a = 1$ we have

$$\lambda_1 = (13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 14)$$

and

$$\lambda_2 = (13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 27, 0)$$

both of which have dimension $2^{94} * 7^{103} * 13 * 17 * 19 * 29 * 31 * 43 * 47 * 61 * 71 * 113 * 127 * 197 * 211$.
For $a = 2$ in A_{15} we have

$$\lambda_1 = (237, 237, 237, 237, 237, 237, 237, 237, 237, 237, 237, 237, 237, 237, 479)$$

$$\lambda_2 = (237, 237, 237, 237, 237, 237, 237, 237, 237, 237, 237, 237, 237, 713, 3)$$

both of which have dimension $2^{124} * 7^{103} * 17^{105} * 19 * 23 * 29 * 37 * 53 * 139 * 149 * 167 * 179 * 199 * 239 * 359 * 953 * 1549 * 1787$

An interesting consequence of this formulation is that the λ_1 and λ_2 can certainly be chosen to be non-dual (since the first and last coordinates always can be chosen in a way such that they differ). Also, the above formulation allows us to find infinitely many repeated dimensions without using the trick from theorem 1.

2.1.2 B_n, $n \geq 2$

We will use the dimension formula for B_n computed in [4] (denoted $\mathfrak{so}(2n+1)$ in that paper):

$$\begin{aligned} \dim V_\lambda &= \prod_{1\leq i<j\leq n} \left(1 + \frac{a_i + \ldots + a_{j-1} + 2(a_j + \ldots + a_{n-1}) + a_n}{2n+1-i-j}\right) \\ &\times \prod_{1\leq i<j\leq n} \left(1 + \frac{a_i + \ldots + a_{j-1}}{j-i}\right) \\ &\times \prod_{1\leq i\leq n} \left(1 + \frac{2(a_i + \ldots + a_{n-1}) + a_n}{2n+1-2i}\right) \end{aligned} \tag{9}$$

Proposition 1. *Choose $n \in \mathbb{N}$ such that $n \geq 3$, and let $\lambda = (a, b, 0, 0, \ldots, 0)$ for some $a, b \in \mathbb{Z}$. Then*

$$\dim V_\lambda = \frac{(a+1)(2a+2b+2n-1)(a+2b+2n-2)(2b+2n-3)(a+b+2n-3)!(b+2n-4)!}{(a+b+1)!b!(2n-3)!(2n-1)!}$$

Proof. For the first part of 9 we have

$$\prod_{1\leq i<j\leq n} \left(1 + \frac{a_i + \ldots + a_{j-1} + 2(a_j + \ldots + a_{n-1}) + a_n}{2n+1-i-j}\right)$$

$$= \left(1 + \frac{a+2b}{2n-2}\right) \times \left(\prod_{i=1,2<j\leq n} \left(1 + \frac{a+b}{2n-j}\right)\right) \times \left(\prod_{i=2<j\leq n-1} \left(1 + \frac{b}{2n-1-j}\right)\right)$$

$$= \frac{2n-2+a+2b}{2n-2} \frac{(2n+a+b-3)!}{(n+a+b-1)!} \frac{(n-1)!}{(2n-3)!} \frac{(2n+b-4)!}{(n-2+b)!} \frac{(n-2)!}{(2n-4)!}$$

Next for the second part we have

$$\prod_{1\le i<j\le n}\left(1+\frac{a_i+\ldots+a_{j-1}}{j-i}\right)$$

$$= (1+a)\times\left(\prod_{i=1,2<j\le n}\left(1+\frac{a+b}{j-1}\right)\right)\times\left(\prod_{i=2<j\le n}\left(1+\frac{b}{j-2}\right)\right)$$

$$= (1+a)\frac{(a+b+n-1)!}{(a+b+1)!}\frac{1}{(n-1)!}\frac{(b+n-2)!}{b!}\frac{1}{(n-2)!}$$

and for the third

$$\prod_{1\le i\le n}\left(1+\frac{2(a_i+\ldots+a_{n-1})+a_n}{2n+1-2i}\right)=\left(1+\frac{2a+2b}{2n-1}\right)\times\left(1+\frac{2b}{2n-3}\right)$$

$$=\frac{2n+2a+2b-1}{2n-1}\frac{2n+2b-3}{2n-3}$$

Thus 9 becomes

$$\frac{2n-2+a+2b}{2n-2}\frac{(2n+a+b-3)!}{(n+a+b-1)!}\frac{(n-1)!}{(2n-3)!}\frac{(2n+b-4)!}{(n-2+b)!}\frac{(n-2)!}{(2n-4)!}$$

$$\times\frac{1+a}{1}\frac{(a+b+n-1)!}{(a+b+1)!}\frac{1}{(n-1)!}\frac{(b+n-2)!}{b!}\frac{1}{(n-2)!}$$

$$\times\frac{2n+2a+2b-1}{2n-1}\frac{2n+2b-3}{2n-3}$$

$$=\frac{(a+1)(2n+2a+2b-1)(a+2b+2n-2)(2b+2n-3)(a+b+2n-3)!(b+2n-4)!}{(a+b+1)!b!(2n-3)!}$$

$$\times\frac{(n-2)!(n-1)!(a+b+n-1)!(b+n-2)!}{(2n-2)(n+a+b-1)!(b+n-2)!(2n-4)!(n-1)!(n-2)!(2n-1)(2n-3)}$$

$$=\frac{(a+1)(2n+2a+2b-1)(a+2b+2n-2)(2b+2n-3)(a+b+2n-3)!(b+2n-4)!}{(a+b+1)!b!(2n-3)!}$$

$$\times\frac{1}{(2n-2)(2n-4)!(2n-1)(2n-3)}$$

$$=\frac{(a+1)(2n+2a+2b-1)(a+2b+2n-2)(2b+2n-3)(a+b+2n-3)!(b+2n-4)!}{(a+b+1)!b!(2n-3)!(2n-1)!}$$

which is precisely what we wanted to show. □

Note that we needed $n \geq 3$ in order to make the first part of 9 work (else the product would involve b instead of $2b$). When $n = 2$ the dimension formula is given by

$$\dim V_{\lambda+1} = \frac{1}{6} ab(a+b)(a+2b)$$

for which (2,1) and (4,0) both have dimension 35. Now consider $\lambda_1 = (0, b, 0, \ldots, 0)$ and $\lambda_2 = (b, a-b, 0, 0, \ldots, 0)$. Then from 1 we have

$$\dim V_{\lambda_1} = \frac{(2n+2b-1)(2b+2n-2)(2b+2n-3)(b+2n-3)!(b+2n-4)!}{(b+1)!b!(2n-3)!(2n-1)!}$$

$$\dim V_{\lambda_2} = \frac{(a+1)(2b+2n-1)(2b-a+2n-2)(2b-2a+2n-3)(b+2n-3)!(b-a+2n-4)!}{(b+1)!(b-a)!(2n-3)!(2n-1)!}$$

so $\dim V_{\lambda_1} = \dim V_{\lambda_2}$ when

$$\frac{(2n+2b-1)(2b+2n-2)(2b+2n-3)(b+2n-3)!(b+2n-4)!}{(b+1)!b!(2n-3)!(2n-1)!}$$

$$= \frac{(a+1)(2b+2n-1)(2b-a+2n-2)(2b-2a+2n-3)(b+2n-3)!(b-a+2n-4)!}{(b+1)!(b-a)!(2n-3)!(2n-1)!}$$

hence

$$\frac{2(b+n-1)(2b+2n-3)(b+2n-4)!}{b!}$$

$$= \frac{(a+1)(2b-a+2n-2)(2b-2a+2n-3)(b-a+2n-4)!}{(b-a)!}$$

This has a large number of solutions, but let us suppose $a = 1$. Then

$$\frac{2(b+n-1)(2b+2n-3)(b+2n-4)!}{b!}$$

$$= \frac{2(2b+2n-3)(2b+2n-5)(b+2n-5)!}{(b-1)!}$$

so

$$\frac{(b+n-1)(b+2n-4)}{b}$$

$$= \frac{2b+2n-5}{1}$$

which has two solutions, namely $b = 2-n$ and $b = 2n-2$, the latter of which will be useful to us (since we want our solution to work for large n).

Thus in B_n, $\lambda_1 = (0, 2n-2, 0, \ldots, 0)$ and $\lambda_2 = (1, 2n-3, 0, \ldots, 0)$ have equal dimension, hence by theorem 1 each B_n admit infinitely many non-isomorphic non-dual irreducible representations of equal dimension.

2.1.3 C_n

The dimension formula for C_n is [4]:

$$\begin{aligned}\dim V_\lambda &= \prod_{1\le i<j\le n}\left(1+\frac{a_i+\ldots+a_{j-1}+2(a_j+\ldots+a_n)}{2n+2-i-j}\right)\\ &\quad\times\prod_{1\le i<j\le n}\left(1+\frac{a_i+\ldots+a_{j-1}}{j-i}\right)\\ &\quad\times\prod_{1\le i\le n}\left(1+\frac{a_i+\ldots+a_n}{n+1-i}\right)\end{aligned}\tag{10}$$

Proposition 2. *Choose $n\in\mathbb{N}$ such that $n\ge 3$, and let $\lambda=(a,b,0,0,\ldots,0)$ for some $a,b\in\mathbb{Z}$. Then*

$$\dim V_\lambda=\frac{(a+2b+2n-1)(a+b+2n-2)!(a+1)(b+2n-3)!}{(a+b+1)!b!(2n-3)!(2n-1)!}$$

Proof. Similar to the first part of 9 we have

$$\prod_{1\le i<j\le n}\left(1+\frac{a_i+\ldots+a_{j-1}+2(a_j+\ldots+a_n)}{2n+2-i-j}\right)$$

$$=\left(1+\frac{a+2b}{2n-1}\right)\times\left(\prod_{i=1,3\le j\le n}\left(1+\frac{a+b}{2n+1-j}\right)\right)\times\left(\prod_{i=2,3\le j\le n}\left(1+\frac{b}{2n-j}\right)\right)$$

$$=\frac{2n+a+2b-1}{2n-1}\frac{(2n+a+b-2)!}{(n+a+b)!}\frac{n!}{(2n-2)!}\frac{(2n+b-3)!}{(n+b-1)!}\frac{(n-1)!}{(2n-3)!}$$

and from the second part of 9 we have

$$\prod_{1\le i<j\le n}\left(1+\frac{a_i+\ldots+a_{j-1}}{j-i}\right)$$

$$=(1+a)\frac{(a+b+n-1)!}{(a+b+1)!}\frac{1}{(n-1)!}\frac{(b+n-2)!}{b!}\frac{1}{(n-2)!}$$

and our third product becomes

$$\prod_{1\le i\le n}\left(1+\frac{a_i+\ldots+a_n}{n+1-i}\right)$$

$$=\left(1+\frac{a+b}{n}\right)\left(1+\frac{b}{n-1}\right)=\frac{n+a+b}{n}\frac{n+b-1}{n-1}$$

Thus 10 becomes

$$\frac{2n+a+2b-1}{2n-1}\frac{(2n+a+b-2)!}{(n+a+b)!}\frac{n!}{(2n-2)!}\frac{(2n+b-3)!}{(n+b-1)!}\frac{(n-1)!}{(2n-3)!}$$

$$\times(1+a)\frac{(a+b+n-1)!}{(a+b+1)!}\frac{1}{(n-1)!}\frac{(b+n-2)!}{b!}\frac{1}{(n-2)!}$$

$$\times\frac{n+a+b}{n}\frac{n+b-1}{n-1}$$

$$=\frac{(a+2b+2n-1)(a+b+2n-2)!(a+1)(b+2n-3)!}{(a+b+1)!b!(2n-3)!(2n-1)!}$$

$$\times\frac{n!(n-1)!)(a+b+n-1)!(b+n-2)!(a+b+n)(b+n-1)}{(n+a+b)!(n+b-1)!n!(n-1)!}$$

$$=\frac{(a+2b+2n-1)(a+b+2n-2)!(a+1)(b+2n-3)!}{(a+b+1)!b!(2n-3)!(2n-1)!}$$

so we have proven our claim. □

The following relation was found by inspecting the known pairs of irreducible representations of C_n of equal dimension and attempting to find a pattern rather than algebraic manipulation, thus we only provide proof that it is correct instead of providing a full derivation.

Proposition 3. *Choose $n \in \mathbb{N}$ such that $n \geq 3$, and let*

$$\lambda_1 = (2n-5,(2n-3)^2,0,0,\ldots,0)$$

$$\lambda_2 = (2n-3,(2n-3)^2-2,0,0,\ldots,0)$$

then

$$\text{dim}V_{\lambda_1} = \text{dim}V_{\lambda_2}$$

Proof. Plugging $a = 2n-5$ and $b = (2n-3)^2$ into proposition 2 we have

$$\text{dim}V_{\lambda_1} = \frac{4(n-2)((2n-3)^2+2n-3)((2n-3)^2+2n-3)!((2n-3)^2+4n-7)!}{(2n-3)!(2n-3)^2!(2n-1)!((2n-3)^2+2n-4)!)}$$

$$= \frac{64(n-2)(n-1)^3n(4n^2-8n+2)!}{(2(n-1))!(2n)!(4(n^2-3n+2))!}$$

and for $a = 2n-3$ and $b = (2n-3)^2 - 2$ we get

$$\dim V_{\lambda_2} = \frac{4(n-1)((2n-3)^2+2n-4)((2n-3)^2+2n-5)!((2n-3)^2+4n-7)!}{(2n-3)!(2n-1)!((2n-3)^2-2)!((2n-3)^2+2n-4)!}$$

$$= \frac{64(n-2)(n-1)^3 n(4n^2-8n+2)!}{(2(n-1))!(2n)!(4(n^2-3n+2))!}$$

hence $\dim V_{\lambda_1} = \dim V_{\lambda_2}$. □

Thus for each $n \in \mathbb{N}$ there exist non-isomorphic and non-dual irreducible representations of C_n of equal dimension with highest weights λ_1, λ_2 as given above.

2.1.4 D_n

From [4] we have

$$\begin{aligned} \dim V_\lambda &= \prod_{1\le i<j\le n-1} \left(1 + \frac{a_i + \ldots + a_{j-1} + 2(a_j + \ldots + a_{n-2}) + a_{n-1} + a_n}{2n-i-j}\right) \\ &\times \prod_{1\le i<j\le n} \left(1 + \frac{a_i + \ldots + a_{j-1}}{j-i}\right) \\ &\times \prod_{1\le i\le n-1} \left(1 + \frac{a_i + \ldots + a_{n-2} + a_n}{n-i}\right) \end{aligned} \tag{11}$$

Proposition 4. *Choose $n \in \mathbb{N}$ such that $n \geq 3$, and let $\lambda = (a, b, 0, 0, \ldots, 0)$ for some $a, b \in \mathbb{Z}$. Then*

$$\dim V_\lambda = \frac{(a+2b+2n-3)(a+b+2n-4)!(a+b+n-1)(a+1)(b+2n-5)!(b+n-2)}{b!(a+b+1)!(n-1)(n-2)(2n-3)!(2n-5)!}$$

Proof. The first product of 11 is

$$\prod_{1\le i<j\le n-1} \left(1 + \frac{a_i + \ldots + a_{j-1} + 2(a_j + \ldots + a_{n-2}) + a_{n-1} + a_n}{2n-i-j}\right)$$

$$= \left(1 + \frac{a+2b}{2n-3}\right) \times \left(\prod_{i=1, 3\le j\le n-1} \left(1 + \frac{a+b}{2n-1-j}\right)\right) \times \left(\prod_{i=2, 3\le j\le n-1} \left(1 + \frac{b}{2n-2-j}\right)\right)$$

$$= \frac{2n+a+2b-3}{2n-3} \frac{(2n+a+b-4)!}{(a+b+n-1)!} \frac{(n-1)!}{(2n-4)!} \frac{(2n+b-5)!}{(2n+b-2)!} \frac{(n-2)!}{(2n-5)!}$$

From our work in B_n we know

$$\prod_{1\le i<j\le n}\left(1+\frac{a_i+\ldots+a_{j-1}}{j-i}\right)$$

$$=(1+a)\frac{(a+b+n-1)!}{(a+b+1)!}\frac{1}{(n-1)!}\frac{(b+n-2)!}{b!}\frac{1}{(n-2)!}$$

and

$$\prod_{1\le i\le n-1}\left(1+\frac{a_i+\ldots+a_{n-2}+a_n}{n-i}\right)$$

$$=\left(1+\frac{a+b}{n-1}\right)(1+\frac{b}{n-2})=\frac{a+b+n-1}{n-1}\frac{b+n-2}{n-2}$$

thus equation 11 reduces to

$$\frac{2n+a+2b-3}{2n-3}\frac{(2n+a+b-4)!}{(a+b+n-1)!}\frac{(n-1)!}{(2n-4)!}\frac{(2n+b-5)!}{(2n+b-2)!}\frac{(n-2)!}{(2n-5)!}$$

$$\times\frac{1+a}{1}\frac{(a+b+n-1)!}{(a+b+1)!}\frac{1}{(n-1)!}\frac{(b+n-2)!}{b!}\frac{1}{(n-2)!}$$

$$\times\frac{a+b+n-1}{n-1}\frac{b+n-2}{n-2}$$

$$=\frac{(a+2b+2n-3)(a+b+2n-4)!(a+b+n-1)(a+1)(b+2n-5)!(b+n-2)}{b!(a+b+1)!(n-1)(n-2)(2n-3)!(2n-5)!}$$

□

Next we will consider $\lambda_1=(0,b,0,\ldots,0)$ and $\lambda_2=(1,b-1,0,\ldots,0)$ and find b as a function of n such that $\dim V_{\lambda_1}=\dim V_{\lambda_2}$.

$$\dim V_{\lambda_1}=\frac{(2b+2n-3)(b+2n-4)!(b+n-1)(b+n-2)(b+2n-5)!}{b!(b+1)!(n-1)(n-2)(2n-3)!(2n-5)!}$$

$$\dim V_{\lambda_2}=\frac{2(2b+2n-4)(b+2n-4)!(b+n-1)(b+n-3)(b+2n-6)!}{(b-1)!(b+1)!(n-1)(n-2)(2n-3)!(2n-5)!}$$

so $\dim V_{\lambda_1}=\dim V_{\lambda_2}$ when

$$\frac{(2b+2n-3)(b+2n-5)!}{b!}$$

$$=\frac{4(b+n-3)(b+2n-6)!}{(b-1)!}$$

so

$$\frac{(2b+2n-3)(b+2n-5)}{b} = 4(b+n-3)$$

has two solutions, $b = 2n-3$ and $b = \frac{5}{2} - n$, the former of which we will use. Thus if

$$\lambda_1 = (0, 2n-3, 0, 0, \dots, 0)$$
$$\lambda_2 = (1, 2n-4, 0, 0, \dots, 0)$$

then

$$\dim V_{\lambda_1} = \dim V_{\lambda_2}$$

2.2 Summary of Results for Infinite Lie Algebra Families

For the reader's convenience we restate, for each compact simple Lie algebra (except type A_1) of type A, B, C, or D, λ_1 and λ_2 such that $\dim V_{\lambda_1} = \dim V_{\lambda_2}$:

1. A_n
   (a) $n = 2$
   $$\lambda_1 = (1, 2)$$
   $$\lambda_2 = (4, 0)$$
   (b) $n \geq 3$. For simplicity we will only mention the $a = 1$ case of equation 8.
   $$\lambda_1 = (n-2, n-2, \dots, n-2, n-2, n-1)$$
   $$\lambda_2 = (n-2, n-2, \dots, n-2, 2n-3, 0)$$
2. B_n, $n \geq 3$
   $$\lambda_1 = (0, 2n-2, 0, \dots, 0)$$
   $$\lambda_2 = (1, 2n-3, 0, \dots, 0)$$
3. C_n, $n \geq 3$
   $$\lambda_1 = (2n-5, (2n-3)^2, 0, 0, \dots, 0)$$
   $$\lambda_2 = (2n-3, (2n-3)^2 - 2, 0, 0, \dots, 0)$$
4. D_n, $n \geq 3$
   $$\lambda_1 = (0, 2n-3, 0, 0, \dots, 0)$$
   $$\lambda_2 = (1, 2n-4, 0, 0, \dots, 0)$$

2.3 Exceptional Algebras

The following section is essentially a summary of pairs of irreducible representations of exceptional Lie algebras of equal dimension as found by [5]. For each exceptional algebra we need only demonstrate the existence of a single pair of non-dual non-isomorphic irreducible representations of equal dimension to demonstrate the existence of infinitely many such pairs, so a full derivation as in the case of the infinite families is unnecessary.

The dimension formula for the exceptional algebras of type E_n will be omitted due to its unwieldly length (for example, the dimension formula for E_8 is a 120th order polynomial in 8 variables).

2.3.1 E_6

The basis for E_6 used by [5] is provided below:

λ	$\dim V_\lambda$
100000	27
010000	7371
001000	2925
000100	351
000010	27
000001	78

Then (000100) and (000020) are non-dual and both have dimension 351.

2.3.2 E_7

The basis for E_7 used is

λ	$\dim V_\lambda$
1000000	133
0100000	8645
0010000	365750
0001000	27664
0000100	1539
0000010	56
0000001	912

There exist two non-isomorphic irreducible representations of dimension 1903725824 with highest weights (0000023) and (0001100) [1].

2.3.3 E_8

The basis used for E_8 is the one used in [6]:

λ	$\dim V_\lambda$
10000000	6696000
01000000	3875
00100000	2450240
00010000	248
00001000	6899079264
00000100	147250
00000010	30380
00000001	146325270

Then the smallest dimension shared by two non-isomorphic irreducible representations of E_8 is 8634368000 [5], shared by $(1,1,0,0,0,0,0,0)$ and $(0,1,0,1,0,0,1,0)$ [7].

2.3.4 F_4

The dimension formula for F_4 is [4]:

$$\begin{aligned}\dim V_\lambda = {} & (1+a_1)(1+a_2)(1+a_3)(1+a_4)(1+\frac{a_1+a_2}{2})(1+\frac{a_2+a_3}{2})(1+\frac{a_3+a_4}{2}) \\ & \times(1+\frac{a_1+a_2+a_3}{3})(1+\frac{a_2+a_3+a_4}{3})(1+\frac{2a_2+a_3}{3})(1+\frac{2a_2+a_3+a_4}{4})(1+\frac{a_1+a_2+a_3+a_4}{4}) \\ & \times(1+\frac{a_1+2a_2+a_3}{4})(1+\frac{2a_2+2a_3+a_4}{5})(1+\frac{a_1+2a_2+a_3+a_4}{5})(1+\frac{2a_1+2a_2+a_3}{5}) \\ & \times(1+\frac{a_1+2a_2+2a_3+a_4}{6})(1+\frac{2a_1+2a_2+a_3+a_4}{6})(1+\frac{a_1+3a_2+2a_3+a_4}{7}) \\ & \times(1+\frac{2a_1+2a_2+2a_3+a_4}{7})(1+\frac{2a_1+3a_2+2a_3+a_4}{8})(1+\frac{2a_1+4a_2+2a_3+a_4}{9}) \\ & \times(1+\frac{2a_1+4a_2+3a_3+a_4}{10})(1+\frac{2a_1+4a_2+3a_3+2a_4}{11})\end{aligned}$$

Then the dimension of (1,0,0,0) is 52, (0,1,0,0) is 1274, (0,0,1,0) is 273, and (0,0,0,1) is 26, consistent with the basis used by [5]. We find that (2,0,0,0) and (1,0,0,1) both have dimension 1053.

2.3.5 G_2

The dimension formula for G_2 is [3]:

$$\dim V_\lambda = \frac{1}{5!}(a_1+1)(a_2+1)(a_1+a_2+2)(a_1+2a_2+3)(a_1+3a_2+4)(2a_1+3a_2+5)$$

It is trivial to verify that $(0,2)$ and $(3,0)$ both have dimension 77.

2.4 Semisimple Algebras

We generalize our result of simple Lie algebras to semisimple ones with the following theorem.

Theorem 4. *Any compact semisimple Lie algebra of rank ≥ 2 admits infinitely many pairs of irreducible representations of equal dimension.*

Proof. Let $\mathfrak{g}$ be a compact semisimple Lie algebra of rank at least 2, and let V_λ denote the irreducible representation of $\mathfrak{g}$ with highest weight λ. Since $\mathfrak{g}$ is semisimple, it decomposes into a direct sum of simple Lie algebras $\mathfrak{g}_\mathfrak{i}$,

$$\mathfrak{g} = \bigoplus_{i=1}^{n} \mathfrak{g}_\mathfrak{i}$$

for some $n \in \mathbb{N}$. Hence we have the following decomposition of irreducible representations in $\mathfrak{g}$:

$$V_\lambda = V_{\lambda_1} \otimes V_{\lambda_2} \otimes \cdots \otimes V_{\lambda_n}$$

where V_{λ_i} denotes the irreducible representation of highest weight λ_i in $\mathfrak{g}_\mathfrak{i}$. From the Weyl dimension formula we have

$$\dim V_\lambda = \prod_{i=1}^{n} \dim V_{\lambda_i}$$

Next we consider two possible cases of what $\mathfrak{g}$ could be.

Case 1. *Suppose that there exists some $j \in \mathbb{N}$ such that the simple Lie algebra $\mathfrak{g}_\mathfrak{j} \in \{\mathfrak{g}_1, \ldots, \mathfrak{g}_\mathfrak{n}\}$ is not $\mathfrak{sl}(2)$ (or equivalently that $\mathfrak{g}$ is not the direct sum of n copies of $\mathfrak{sl}(2)$). Setting λ_i to zero for all $i \neq j$ we have*

$$\dim V_\lambda = \dim V_{\lambda_j} \prod_{i \neq j}^{n} \dim V_{\lambda_i} = \dim V_{\lambda_j}$$

where $V_{\lambda_{i\neq j}}$ is the trivial representation in $\mathfrak{g}_\mathfrak{i}$ of dimension 1. Then, by Theorem 3 there exist infinitely many pairs of non-isomorphic irreducible representations of $\mathfrak{g}_\mathfrak{j}$ of equal dimension. But for each of these pairs of irreducible representations of g_j, labeled by λ_{j_1} and λ_{j_2}, we have an irreducible representation of $\mathfrak{g}$ given by (to minimize confusion here we will label the irreducible representations of $\mathfrak{g}$ by λ_a and λ_b)

$$V_{\lambda_a} = V_{\lambda_1=0} \otimes V_{\lambda_2=0} \otimes \cdots \otimes V_{\lambda_j-1=0} \otimes V_{\lambda_{j_1}} \otimes V_{\lambda_{j+1}=0} \cdots \otimes V_{\lambda_n=0}$$

and

$$V_{\lambda_b} = V_{\lambda_1=0} \otimes V_{\lambda_2=0} \otimes \cdots \otimes V_{\lambda_j-1=0} \otimes V_{\lambda_{j_2}} \otimes V_{\lambda_{j+1}=0} \cdots \otimes V_{\lambda_n=0}$$

where

$$\dim V_{\lambda_a} = \dim V_{\lambda_{j_1}} = \dim V_{\lambda_{j_2}} = \dim V_{\lambda_b}$$

Since we have infinitely many λ_{j_1} and λ_{j_2} of equal dimension in $\mathfrak{g}_\mathfrak{j}$, we have infinitely many λ_a and λ_b of equal dimension in $\mathfrak{g}$.

Case 2. *Suppose that there exists no* $j \in \mathbb{N}$ *such that the simple Lie algebra* $\mathfrak{g_j} \in \{\mathfrak{g_1}, \ldots, \mathfrak{g_n}\}$ *is not* $\mathfrak{sl}(2)$ *(or equivalently that* $\mathfrak{g}$ *is the direct sum of* n *copies of* $\mathfrak{sl}(2)$*). Because* $\mathfrak{g}$ *is of rank* ≥ 2*, we know that* $n \geq 2$*. Because an irreducible representation of* $sl(2)$ *with highest weight* λ *has dimension* $\lambda + 1$ *we know that for all* $k \in \mathbb{N}$ *the irreducible representation of* $\mathfrak{g}$ *given by the choice of* $\lambda_1 = 6k - 1$*,* $\lambda_2 = k - 1$ *(and all other* $\lambda_i = 0$*) will have dimension* $6k^2$*, as will the irreducible representation of* $\mathfrak{g}$ *given by the choice of* $\lambda_1 = 3k - 1$*,* $\lambda_2 = 2k - 1$ *(and all other* $\lambda_i = 0$*). That is, for irreducible representations* V_{λ_a} *and* V_{λ_b} *of* $\mathfrak{g}$ *given by*

$$V_{\lambda_a} = V_{\lambda_1 = 6k-1} \otimes V_{\lambda_2 = k-1} \otimes V_{\lambda_3 = 0} \cdots \otimes V_{\lambda_n = 0}$$

$$V_{\lambda_b} = V_{\lambda_1 = 3k-1} \otimes V_{\lambda_2 = 2k-1} \otimes V_{\lambda_3 = 0} \cdots \otimes V_{\lambda_n = 0}$$

we have

$$\dim V_{\lambda_a} = (6k - 1 + 1)(k - 1 + 1)(1) \ldots (1) = 6k^2$$
$$\dim V_{\lambda_b} = (3k - 1 + 1)(2k - 1 + 1)(1) \ldots (1) = 6k^2$$

hence $\dim V_{\lambda_a} = \dim V_{\lambda_b}$*. Since this is true for all* $k \in \mathbb{N}$*, then, we have constructed infinitely many pairs of non-isomorphic irreducible representations of* $\mathfrak{g}$ *of equal dimension.*

□

3 Special Properties of A_2

Here we state the theorem that will be proven by the end of this section.

Theorem 5. *For any* $k \in \mathbb{N}$ *there exist infinitely many sets of* k *non-dual non-isomorphic irreducible representations of* A_2 *with equal dimension.*

To approach the proof for this we will reintroduce the homogenized dimension function for A_2:

$$f : \mathbb{R}^2 \mapsto \mathbb{R}$$

$$f(x, y) = xy(x + y)$$

where

$$\dim V_\lambda = \frac{1}{2} f(a + 1, b + 1)$$

for $\lambda \in \mathbb{Z}^2$. Recall that theorem 2 means that we can look for rational arguments of f with equal value c, and moreover our particular choice of c doesn't actually matter so long as there exists some irreducible representation of A_2 with dimension equal to $\frac{c}{2}$. For computational simplicity we will choose $c = 6$. Recall that theorem 1 tells us that if we find m solutions to $f(x, y) = c$ then we obtain infinitely many sets of m irreducible representations with equal dimension.

As a final note, sometimes the below solutions may run negative, but observe that sending $(x, y) \to (x, -x-y)$ or $(x, y) \to (-x-y, y)$ does not change the value of equation $xy(x+y)$ and allows us to choose strictly positive solutions.

We will construct our elliptic curve using largely the same process as in [8]. We start with a homogeneous cubic curve $\mathcal{E}$ over $\mathbb{Q}$ given by

$$X^2Y + XY^2 - 6Z^3 = 0 \tag{12}$$

Note that by setting $Z = 1$ we obtain our usual f. Then $\mathcal{E}$ contains the point $P = (2 : 1 : 1)$. The tangent line l of a function F at a point $P = (P_X : P_Y : P_Z)$ is given by

$$\frac{\partial F}{\partial X}(P)(X - P_X) + \frac{\partial F}{\partial Y}(P)(Y - P_Y) + \frac{\partial F}{\partial Z}(P)(X - P_Z) = 0$$

so at $P = (2 : 1 : 1)$ we have

$$l : 5X + 8Y - 18Z = 0$$

using a tangent-chord construction, we find another rational point Q, the intersection of l and $\mathcal{E}$, given by $Q = (-\frac{32}{5} : \frac{25}{4} : 1)$. The tangent m of $\mathcal{E}$ at Q is computed:

$$m : -\frac{655}{16}X - \frac{976}{25}Y - 18Z = 0$$

We obtain a third rational point R by finding the intersection of m and $\mathcal{E}$, given by $R = (\frac{476288}{165715} : -\frac{429025}{123464} : 1)$. After rescaling our points to be integers, we consider the matrix

$$M_\gamma = \begin{bmatrix} P_X & Q_X & R_X \\ P_Y & Q_Y & R_Y \\ P_Z & Q_Z & R_Z \end{bmatrix} = \begin{bmatrix} 2 & -128 & 232428544 \\ 1 & 125 & -281011375 \\ 1 & 20 & 80868920 \end{bmatrix}$$

Note that M_γ defines the transformation $\gamma : (X, Y, Z) \to (2U - 128V + 232428544W, U + 125V - 281011375W, U + 20V + 80868920W)$. Also, observe that $\det M_\gamma = 53373365640$, hence M_γ is invertible. Note that $M_\delta := M_\gamma^{-1}$ maps $P \to (1 : 0 : 0)$, $Q \to (0 : 1 : 0)$, and $R \to (0 : 0 : 1)$. Applying γ to $\mathcal{E}$ gives us

$$-55566UV^2 - 2541588840U^2W + 131905126836UVW - 297648654602782014UW^2 - 5317378991792784000VW^2$$

Dividing by 55566 we get

$$-UV^2 - 45740U^2W + 2373846UVW - 5356668729129UW^2 - 95694831224000VW^2$$

Then applying a change of variables $(U : V : W) \to (K^2 : LN : KN)$ and dividing by K^2N we obtain

$$-45740K^3 - 5356668729129K^2N + 2373846KLN - L^2N - 95694831224000LN^2$$

Dividing by -45740 we get

$$K^3 + \frac{5356668729129}{45740}K^2N - \frac{1186923}{22870}KLN + \frac{1}{45740}L^2N + 2092147600LN^2$$

and applying a substitution of $N = -45740M$ we get

$$K^3 - 5356668729129K^2M + 2373846KLM - L^2M + 43770815801857600000LM^2$$

Setting $M = 1$ and relabeling K by x, L by y, we obtain the elliptic curve

$$y^2 - 2373846xy - 43770815801857600000y = x^3 - 5356668729129x^2 \tag{13}$$

Note that equation 13 is a Weierstrass equation, and thus admits a minimal model via an isomorphism [9] of the form

$$x = u^2x' + r$$

$$y = u^3y' + su^2x' + t$$

for some $u, r, s, t \in \mathbb{Q}$. Thus we can construct an isomorphism ϕ (i.e. rational group preserving map) by choosing

$$(u, r, s, t) = (\frac{1}{960540}, -\frac{629}{441}, -\frac{395641}{320180}, -\frac{22870}{9261})$$

whose inverse ϕ^{-1} can be obtained by choosing

$$(u, r, s, t) = (960540, 1315960840400, 1186923, 3750484978662969200)$$

Applying ϕ to both sides of of equation 13 and dividing by 785403315065534557045288735296000000 we get

$$y^2 = x^3 + 9 \tag{14}$$

We claim that equation 14 is the minimal model of the curve defined by 13; however, this fact is not crucial so we will not prove it. For the purposes of this section of the paper, all that we need to know about equation 14 is that it retains the same rational group structure as 13 due to properties of isomorphisms of elliptic curves. Computing the rank of this curve using Sage, we have

```
SAGE: R.<x,y>=QQ[]
SAGE: E = EllipticCurve(y^2-x^3-9)
SAGE: E.rank()

1
```

Thus our elliptic curve has rank 1, hence it contains infinitely many rational points. Because all steps between equations 12 and 14 were invertible and preserved rational structure, this means that equation 12 has infinitely many rational solutions (at $Z = 1$), completing our proof of theorem 5.

3.1 Examples

In this section we will give a demonstration in Sage of the above construction as well as producing several rational solutions to $x^2y+xy^2=6$. To do so we will have Sage compute a birational transformation from equation 12 to an elliptic curve (which we observe to be the same as 13).

```
SAGE: R.<x,y,z> = QQ[]
SAGE: cubic = x^2*y+x*y^2-6*z^3
SAGE: P = [2,1,1]
SAGE: f = EllipticCurve_from_cubic(cubic,P,morphism=True)
SAGE: f

Scheme morphism:
  From: Closed subscheme of Projective Space of dimension 2 over Rational
        Field defined by:
  x^2*y + x*y^2 - 6*z^3
  To:   Elliptic Curve defined by y^2 - 2373846*x*y -
       437708158018576000*y = x^3 - 5356668729129*x^2 over Rational
       Field
  Defn: Defined on coordinates by sending (x : y : z) to
        (-16375/28245185096688*x^2 - 5227/3530648137086*x*y -
            7808/8826620342715*y^2 + 2875/1569176949816*x*z +
            1552/980735593635*y*z + 20/21794124303*z^2 :
            -56436093625/28245185096688*x^2 - 4052787479/1765324068543*x*
            y - 3285434624/8826620342715*y^2 + 5504551225/1569176949816*x
            *z + 3934252096/980735593635*y*z + 42034460/21794124303*z^2 :
             -1/1181861924231831342976*x^2 - 1/369331851322447294680*x*
            y - 1/461664814153059118350*y^2 + 1/164147489476643242080*x
            *z + 1/102592180922902026300*y*z - 1/911930497092462456000*z
            ^2)
```

Thus f is a birational transformation from rational solutions of

$$x^2y+xy^2-6z^3 \tag{15}$$

to rational solutions of the elliptic curve

$$y^2-2373846xy-437708158018576000y=x^3-5356668729129x^2 \tag{16}$$

Next, we are concerned with the inverse transformation of f, which is also birational.

```
SAGE: finv = f.inverse(); finv

Scheme morphism:
  From: Elliptic Curve defined by y^2 - 2373846*x*y -
       437708158018576000*y = x^3 - 5356668729129*x^2 over Rational
       Field
  To:   Closed subscheme of Projective Space of dimension 2 over Rational
        Field defined by:
```

```
  x^2*y + x*y^2 - 6*z^3
  Defn: Defined on coordinates by sending (x : y : z) to
        (2*x^2 - 10631281602560*x*z + 5854720*y*z : x^2 + 12853460292500*
            x*z - 5717500*y*z : x^2 - 3698944400800*x*z - 914800*y*z)
```

thus rational solutions to equation 13 can be transformed to rational solutions to equation 15 by

$$
\begin{aligned}
(x, y, z) \rightarrow & (2x^2 - 10631281602560 * x * z + 5854720 * y * z, \\
& x^2 + 12853460292500 * x * z - 5717500 * y * z, \\
& x^2 - 3698944400800 * x * z - 914800 * y * z)
\end{aligned}
$$

Next we compute some properties of our elliptic curve. As in the previous section, we check its rank.

```
SAGE: x,y = polygens(QQ,'x,y')
SAGE: E = EllipticCurve(y^2 - 2373846*x*y - 4377081580185760000*y - x^3 +
    5356668729129*x^2)
SAGE: E.rank()

1
```

its minimal model

```
SAGE: M = E.minimal_model(); M

Elliptic Curve defined by y^2 = x^3 + 9 over Rational Field
```

and as a sanity check, its minimal model's rank

```
SAGE: M.rank()

1
```

To find the isomorphism ϕ from equation 13 to equation 14 we used the Sage command `isomorphisms`, which returns `(u,r,s,t)` in the form used in the previous section. Note that to obtain equation 14 we still need to divide by 785403315065534557045288735296000000.

```
SAGE: from sage.schemes.elliptic_curves.weierstrass_morphism import *
SAGE: isomorphisms(M,E)[1]

(1/960540, -629/441, -395641/320180, -22870/9261)
```

with an inverse ϕ^{-1} found by

```
SAGE: isomorphisms(E,M)[1]

(960540, 1315960840400, 1186923, 3750484978662969200)
```

Because the numbers involved are substantially smaller for M than for E, we will compute rational points of M (and note that they correspond to rational points of E via ϕ^{-1}.

Our elliptic curve M has nonzero rank, so we would expect to be able to find a generator:

```
SAGE: P = M.gens()[0];P
(-2 : 1 : 1)
```

Moreover, because M has rank exactly 1 we expect there to be an error if we try finding more than one generator:

```
SAGE: P = M.gens()[1];P

Error in lines 1-1
Traceback (most recent call last):
  File "/cocalc/lib/python2.7/site-packages/smc_sagews/sage_server.py",
      line 1044, in execute
    exec compile(block+'\n', '', 'single', flags=compile_flags) in
        namespace, locals
  File "", line 1, in <module>
IndexError: list index out of range
```

Next we compute some rational points on M. Note that we can compute as (countably) many rational points of M as we like; we provide as many as will fit on the page.

```
SAGE: for i in range(1,7):
SAGE:   print(P*i)

(-2 : 1 : 1)
(40 : -253 : 1)
(-629/441 : 22870/9261 : 1)
(639280/64009 : -513439919/16194277 : 1)
(-181479482/333756361 : 18128073165931/6097394959109 : 1)
(4040707888729/922637091600 : -8546494108076860067/886229831965464000 : 1)
```

We can check that these are indeed solutions of M by computing $y^2 - x^3 - 9$:

```
SAGE: for i in range(1,7):
SAGE:     x = (P*i)[0]
SAGE:     y = (P*i)[1]
SAGE:     print(y^2-x^3-9)

0
0
0
0
0
0
```

Using ϕ^{-1} we transform these rational solutions to M to rational solutions of E:

```
SAGE: (u,r,s,t) = (960540, 1315960840400, 1186923, 3750484978662969200)
SAGE:
SAGE: for i in range(1,7):
SAGE:     a = (P*i)[0]
SAGE:     b = (P*i)[1]
SAGE:     x = (u^2 *a + r)
SAGE:     y = (u^3 *b + s* u^2* a + t)
SAGE:     print(x,y)

(-529313342800, 2446516441282139600)
(38221444504400, -176661695121673550800)
(0, 4377081580185760000)
(674056777351211600/64009, -2171704037961283646552356OO/16194277)
(271770599852851233200/333756361, 3530308231808242532059430203240O/6097394959109)
(5356668729129, 0)
```

Checking that these are in fact points of E we compute $y^2 - 2373846xy - 4377081580185760000y - x^3 + 5356668729129x^2$:

```
SAGE: (u,r,s,t) = (960540, 1315960840400, 1186923, 3750484978662969200)
SAGE:
SAGE: for i in range(1,7):
SAGE:    a = (P*i)[0]
SAGE:    b = (P*i)[1]
SAGE:    x = (u^2 *a + r)
SAGE:    y = (u^3 *b + s* u^2* a + t)
SAGE:    print(y^2 - 2373846*x*y - 4377081580185760000*y - x^3 +
    5356668729129*x^2)

0
0
0
0
0
0
```

As one would probably anticipate, ϕ^{-1} applied to the generators of M simply produces the generators of E:

```
SAGE: Q = E.gens()[0]
SAGE: for i in range(1,7):
SAGE:    print Q*i

(-529313342800 : 2446516441282139600 : 1)
(38221444504400 : -176661695121673550800 : 1)
(0 : 4377081580185760000 : 1)
(674056777351211600/64009 : -217170403796128364655235600/16194277 : 1)
```

```
(271770599852851233200/333756361 :
    35303082318082425320594302032400/6097394959109 : 1)
(5356668729129 : 0 : 1)
```

Finally, we invert the morphism (denoted by `finv`) used to go from equation 15 to 16:

```
SAGE: for i in range(1,7):
SAGE:     print(finv(P*i))

(-1 : 1 : 0)
(1 : 2 : 1)
(-32/5 : 25/4 : 1)
(289/777 : 1369/357 : 1)
(-429025/123464 : 476288/165715 : 1)
(23791202/480639121 : 692163481/63009781 : 1)
```

We conclude our example by showing that these solutions actually lie on the curve $x^2y + xy^2 = 6z^3$. Note that the first is a trivial ($z = 0$) solution, but all others solve equation 15 for $z = 1$.

```
SAGE: for i in range(1,7):
SAGE:     x = finv(P*i)[0]
SAGE:     y = finv(P*i)[1]
SAGE:     print(x^2*y + x*y^2)

0
6
6
6
6
6
```

4 Other Rank 2 Simple Lie Algebras

We will demonstrate that the above results of A_2 do not generalize to the other rank 2 algebras, namely B_2 and G_2. Our proof will largely hinge on a theorem, originally conjectured by Mordell, proven by Faltings:

Theorem 6 (Faltings)**.** *Let C be an algebraic curve of genus $g \geq 2$. Then the set of rational solutions $C(Q)$ is finite.*

For a proof of this we refer the reader to [10]. For nonsingular plane curves we have a convenient way to compute g, given by [11]:

Theorem 7. *Let C be a nonsingular plane curve of degree n, then*

$$g = \frac{(n-1)(n-2)}{2}$$

Recall that for $\lambda = (x, y)$,

$$f_{B_2}(\lambda) = \frac{(2x+y)(x+y)xy}{6}$$

$$f_{G_2}(\lambda) = \frac{xy(x+y)(x+2y)(x+3y)(2x+3y)}{120}$$

Then the the degree of f_{B_2} is 4, hence it has genus $\frac{3*2}{2} = 3$. The degree of f_{G_2} is 6, hence it has genus $\frac{5*4}{2} = 10$. Then by Faltings' theorem there exists an integer N_{B_2} such for $(a, b) \in \mathbb{Z}^2$, $a, b \geq 0$ there are no more than N_{B_2} rational solutions $(x, y) \in \mathbb{Q}^2$ to $f_{G_2}(x, y) = f_{G_2}(a, b)$. It follows that there are no more than N_{B_2} irreducible representations of B_2 of equal dimension. The same argument applied to G_2 demonstrates that there exists some N_{G_2} such that there are no more than N_{G_2} irreducible representations of G_2 of equal dimension.

5 Higher Rank Simple Lie Algebras

Note that Faltings' theorem only concerns algebraic curves, that is, 1 dimensional projective varieties. The Bombieri-Lang conjecture generalizes this. The following has been reproduced from [12]:

Conjecture (Bombieri-Lang)**.** *Let X be a projective variety of general type defined over a number field K. Then the set of rational points X(K) is not Zariski dense in X.*

Note that a smooth hypersurface of degree d in $\mathbb{P}^n$ $(n \geq 3)$ is of general type if $d > n + 1$ [13]. It is trivial to verify that for a simple Lie algebra $\mathfrak{g}$ of rank > 2, f_g is smooth and of dimension $> n + 1$. Then by the same logic as in the previous section, it follows that the Bombieri-Lang conjecture would imply that there exist some number $n_{\mathfrak{g}} \in \mathbb{Z}$ such that there are no more than $n_{\mathfrak{g}}$ non-isomorphic non-dual irreducible representations of $\mathfrak{g}$ of equal dimension.

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